Exercise 27

The advection equation. This is the equation

$$\frac{\partial u}{\partial t} = -\kappa \frac{\partial u}{\partial x}(x,t) + k(x,t), \tag{12}$$

where κ is a positive constant and k is a function of (x,t). It is used in modeling AIDS epidemics, fluid dynamics, and other situations involving matter that is carried along with a flow of air or water. As a concrete application, suppose that the wind is blowing in one direction at the speed of κ meters per second, say, the positive direction on the x-axis, and it is carrying with it a certain pollutant from a factory located at the origin. Let $u(x,t)$ denote the density (number of particles per meter) at time t, and suppose that the particles are falling out of the air at a constant rate proportional to $u(x,t)$, with constant of proportionality $r > 0$. Then u satisfies (12) with $k(x,t) = -ru(x,t)$. (For a derivation and references see Modeling Differential Equations in Biology, by Clifford Henry Taub, Prentice Hall, 2001.)

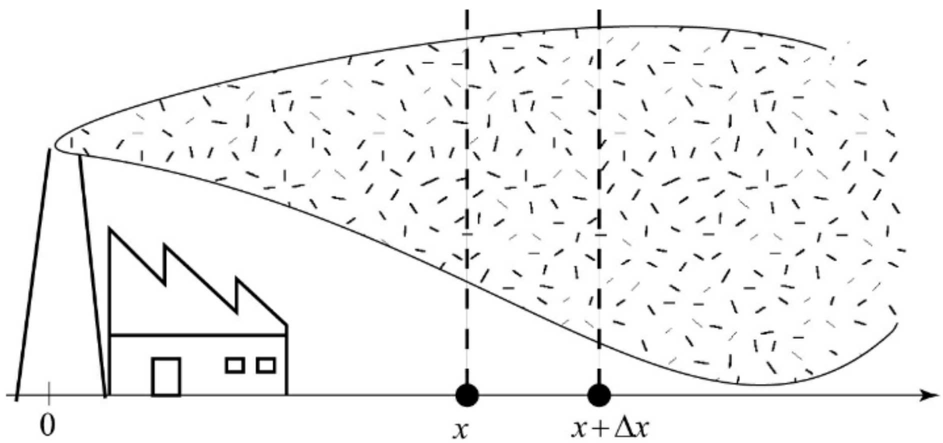

Figure 9 The number of particles at time t between x and $x + \Delta x$ is $u(x,t)\Delta x$.

(a) Show that all the solutions of

$$\frac{\partial u}{\partial t}(x,t) = -\kappa \frac{\partial u}{\partial x}(x,t) - ru(x,t)$$

are of the form $u(x,t) = e^{-rt} f(x - \kappa t)$. [Hint: Use a change of variables as we did in Example 3, Section 1.1.]

(b) Let M denote the number of particles in the air at time $t = 0$. Show that the number of particles at time $t > 0$ is $e^{-rt}M$. [Hint: The number of particles is the integral of the density function over the interval $-\infty < x < \infty$.]

Solution

Part (a)

Start by bringing all terms to the left side.

$$\frac{\partial u}{\partial t} + \kappa \frac{\partial u}{\partial x} + ru(x,t) = 0$$

In order to get the specific formula for u, start by using an integrating factor to combine the terms with u and $\partial u/\partial t$.

$$I = \exp\left(\int^t r\,ds\right) = e^{rt}$$

Multiply both sides of equation (1) by I.

$$e^{rt}\frac{\partial u}{\partial t} + \kappa e^{rt}\frac{\partial u}{\partial x} + re^{rt}u(x,t) = 0$$

Use the product rule to combine the first and third terms.

$$\frac{\partial}{\partial t}(e^{rt}u) + \kappa e^{rt}\frac{\partial u}{\partial x} = 0$$

The partial derivative with respect to x treats t as if it's a constant, so bring e^{rt} inside.

$$\frac{\partial}{\partial t}(e^{rt}u) + \kappa \frac{\partial}{\partial x}(e^{rt}u) = 0 \tag{1}$$

Now make the change of variables, $\alpha = x + \kappa t$ and $\beta = x - \kappa t$, and use the chain rule to write the derivatives in terms of these new variables.

$$\frac{\partial}{\partial x} = \frac{\partial \alpha}{\partial x}\frac{\partial}{\partial \alpha} + \frac{\partial \beta}{\partial x}\frac{\partial}{\partial \beta} = (1)\frac{\partial}{\partial \alpha} + (1)\frac{\partial}{\partial \beta} = \frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \beta}$$

$$\frac{\partial}{\partial t} = \frac{\partial \alpha}{\partial t}\frac{\partial}{\partial \alpha} + \frac{\partial \beta}{\partial t}\frac{\partial}{\partial \beta} = (\kappa)\frac{\partial}{\partial \alpha} + (-\kappa)\frac{\partial}{\partial \beta} = \kappa\frac{\partial}{\partial \alpha} - \kappa\frac{\partial}{\partial \beta}$$

As a result, equation (1) becomes

$$\begin{aligned}
0 &= \frac{\partial}{\partial t}(e^{rt}u) + \kappa\frac{\partial}{\partial x}(e^{rt}u) \\
0 &= \left(\kappa\frac{\partial}{\partial \alpha} - \kappa\frac{\partial}{\partial \beta}\right)e^{rt}u + \kappa\left(\frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \beta}\right)e^{rt}u \\
&= \kappa\frac{\partial}{\partial \alpha}(e^{rt}u) - \kappa\frac{\partial}{\partial \beta}(e^{rt}u) + \kappa\frac{\partial}{\partial \alpha}(e^{rt}u) + \kappa\frac{\partial}{\partial \beta}(e^{rt}u) \\
&= 2\kappa\frac{\partial}{\partial \alpha}(e^{rt}u)
\end{aligned}$$

Divide both sides by 2κ.

$$\frac{\partial}{\partial \alpha}(e^{rt}u) = 0$$

Integrate both sides with respect to α.

$$e^{rt}u = f(\beta)$$

Here f is an arbitrary function. Change back to the original variables now.

$$e^{rt}u = f(x - \kappa t)$$

Therefore, dividing both sides by e^{rt},

$$u(x,t) = e^{-rt}f(x - \kappa t).$$

Part (b)

Start by bringing all terms to the left side.

$$\frac{\partial u}{\partial t} + \kappa\frac{\partial u}{\partial x} + ru(x,t) = 0$$

Integrate both sides with respect to x from $-\infty$ to ∞.

$$\begin{aligned}
\int_{-\infty}^{\infty}(0)\,dx &= \int_{-\infty}^{\infty}\left[\frac{\partial u}{\partial t} + \kappa\frac{\partial u}{\partial x} + ru(x,t)\right]dx \\
0 &= \int_{-\infty}^{\infty}\frac{\partial u}{\partial t}\,dx + \kappa\int_{-\infty}^{\infty}\frac{\partial u}{\partial x}\,dx + r\int_{-\infty}^{\infty}u(x,t)\,dx \\
&= \frac{d}{dt}\left[\int_{-\infty}^{\infty}u(x,t)\,dx\right] + \kappa u(x,t)\Big|_{-\infty}^{\infty} + r\left[\int_{-\infty}^{\infty}u(x,t)\,dx\right] \\
&= \frac{d}{dt}\left[\int_{-\infty}^{\infty}u(x,t)\,dx\right] + \underbrace{\kappa[u(\infty,t) - u(-\infty,t)]}_{=0} + r\left[\int_{-\infty}^{\infty}u(x,t)\,dx\right]
\end{aligned}$$

The pollution particle density far from the origin is zero, so $u(\pm\infty,t) = 0$. Also, the partial time derivative becomes a total derivative in front of the integral because the integral wipes out x, leaving only a function of t.

$$\frac{d}{dt}\left[\int_{-\infty}^{\infty}u(x,t)\,dx\right] + r\left[\int_{-\infty}^{\infty}u(x,t)\,dx\right] = 0$$

This is a first-order ODE for the integral, so multiply both sides by the integrating factor e^{rt}.

$$e^{rt}\frac{d}{dt}\left[\int_{-\infty}^{\infty}u(x,t)\,dx\right] + re^{rt}\left[\int_{-\infty}^{\infty}u(x,t)\,dx\right] = 0$$

Use the product rule to rewrite the left side.

$$\frac{d}{dt}\left[e^{rt}\int_{-\infty}^{\infty}u(x,t)\,dx\right] = 0$$

Integrate both sides with respect to t.

$$e^{rt}\int_{-\infty}^{\infty}u(x,t)\,dx = C$$

This equation holds for all values of t, and the constant C is the same for all of them. Set $t = 0$ to determine C.

$$e^{r(0)} \underbrace{\int_{-\infty}^{\infty} u(x,0)\,dx}_{=\,M} = C$$

$$M = C$$

The previous equation then becomes

$$e^{rt} \int_{-\infty}^{\infty} u(x,t)\,dx = M.$$

Therefore, dividing both sides by e^{rt}, the number of particles at time t is

$$\int_{-\infty}^{\infty} u(x,t)\,dx = Me^{-rt}.$$